Predicative Implications: A Topological Approach

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Therefore, to check that if $f : A \to B$ we have to check the condition $f(a) : B$ *for all* $a : A$, including all $ev(F, g)$ *for all* $g : A \to B$ and all $F : (A \to B) \to A$. Since the quantifier on g also refers to f itself, the definition would be impredicative.

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How to solve the impredicativity?

Exclude modus ponens from the logic and reflexivity condition from the Kripke models. Work with the transitive (serial) persistent Kripke models.

A General Notion of Implication

Definition

Let $(A, \leq, \wedge, 1)$ be a bounded meet-semilatice. By an implication $\rightarrow: A^{op} \times A \Rightarrow A$ we mean any function with the following properties:

(*i*) If $a \leq b$ then $a \rightarrow b = 1$,

(*ii*) $(a \rightarrow b) \wedge (b \rightarrow c) \leq (a \rightarrow c)$,

(*iii*) $(a \rightarrow b) \wedge (a \rightarrow c) \leq (a \rightarrow b \wedge c)$.

If the converse of (i) also holds, i.e. if $a \rightarrow b = 1$ implies $a \leq b$, then the implication is called an internal order. Moreover, the structure $(A, \leq, \wedge, 1, \rightarrow)$ is called a strong algebra if $\rightarrow$ is an implication and a closed algebra if $\rightarrow$ is an internal order.

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Example

For a bounded meet-semilattice A, for all $a, b \in A$ define $a \rightarrow b = 1$. Then $\rightarrow$ is an implication.

Some Examples

Example

Let A be a non-trivial bounded meet-semilattice. Pick $x \neq 1$ and define $a \to_x b = 1$ if $a \leq b$ and otherwise $a \to_x b = x$. Then $\to_x$ is an internal order.

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Let X be a locale and $J : X \to X$ be an increasing, join preserving function. Then the pair (X, J) is called a modal space.

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Example

Let (X, J) be a modal space. Define $\to_J$ as $a \to_J b = \bigvee\{c | Jc \wedge a \leq b\}$, i.e, as the right adjoint in the pair $J(-) \wedge a \dashv a \to_J (-)$. Then $(X, \to)$ is a strong algebra. If $J1 = 1$ the algebra is also closed.

Modal Space Generates an Implication

$$\dfrac{\dfrac{a \leq b}{J1 \wedge a \leq b}}{1 \leq a \to b} \qquad \dfrac{\dfrac{1 \leq a \to b}{J1 \wedge a \leq b}}{a \leq b}\,*$$

* Since $J1 = 1$.

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For internal transitivity we have:

$$\dfrac{\dfrac{\dfrac{\overline{\overline{J(a \to b) \wedge a \leq b}} \qquad \overline{\overline{J(b \to c) \wedge b \leq c}}}{J(a \to b) \wedge J(b \to c) \wedge a \leq c}}{\overline{\overline{J((a \to b) \wedge (b \to c)) \wedge a \leq c}}}}{(a \to b) \wedge (b \to c) \leq a \to c}$$

Kripke Frame as a Modal Space

Example

Assume that (W, R) is a relational frame, i.e., $R \subseteq W \times W$. Pick the discrete topology and define $J : P(W) \to P(W)$ as $J(U) = \{x | \exists y \in U \ R(y, x)\}$. Since J is trivially monotone and join preserving, $(P(W), J)$ is a modal space.

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In case $R \subseteq W \times W$ is transitive it is possible to change $P(W)$ by $UP(W)$, the set of all upsets of W. Then, $((W, UP(W)), J)$ is another modal space arising from R.

The Roles of Modal Spaces: J as a Temporal Operator

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  means that $u \to v$ is provable by w iff the fact that "w happened before" together with u, implies v.
- This time lag makes a delay between introducing an implication, and using it in the applications. For instance, $u \wedge (u \to v)$ does not necessarily imply v, but if $u \to v$ has been proved before, that is if we have $u \wedge J(u \to v)$, then we can prove v.

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- Note that this interpretation also validates $Ja \leq a$ that we do not have in an arbitrary modal space.

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Representation Theorem (A., Alizadeh, Memarzadeh)

If A is a strong algebra then there exists a modal space X such that A is embedable in X as a strong algebra.

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Philosophical Consequence

Any implication is a predicative implication enlarging the domain of the discourse.

Let $\mathcal{L}_J$ be the usual language of propositional logic with a unary modal operator J. Define **mJ** as usual natural deduction rules for all connectives except implication (and hence negation) plus:

Structural Rules:

$$F\ \frac{\Gamma \vdash A}{J\Gamma \vdash JA} \qquad cut\ \frac{\Gamma \vdash A \qquad \Pi, A \vdash B}{\Gamma, \Pi \vdash B}$$

Propositional Rules:

$$\to_E\ \frac{\Gamma \vdash A \qquad \Pi \vdash J(A \to B)}{\Gamma, \Pi \vdash B} \qquad \to_I\ \frac{J\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

Note that in the rules $\to I$ and F, Γ can have exactly one element.

More Predicative Logics

Consider the following rules:

Additional Rules:

$$\frac{JA \vdash \bot}{A \vdash \bot}\;{\scriptstyle sCoJ} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash JA}\;{\scriptstyle CoJ} \qquad \frac{\Gamma \vdash JA}{\Gamma \vdash A}\;{\scriptstyle J}$$

Then define:

- $\mathbf{J} = \mathbf{mJ} + J$
- $\mathbf{CoJ} = \mathbf{mJ} + CoJ$
- $\mathbf{sCoJ} = \mathbf{mJ} + sCoJ$
- $\mathbf{sI} = \mathbf{mJ} + J + sCoJ$

Topological/Kripke Semantics

Definition

A topological model is a tuple (X, J, V) such that (X, J) is a modal space and $V : \mathcal{L}_J \to X$ is a valuation function such that:

(*i*) $V(\top) = 1$ and $V(\bot) = 0$.

(*ii*) $V(A \wedge B) = V(A) \wedge V(B)$.

(*iii*) $V(A \vee B) = V(A) \vee V(B)$.

(*iv*) $V(A \to B) = V(A) \to_J V(B)$.

(*v*) $V(JA) = JV(A)$.

We say $(X, J, V) \vDash \Gamma \Rightarrow A$ when $\bigwedge_{\gamma \in \Gamma} V(\gamma) \leq V(A)$ and $(X, J) \vDash \Gamma \Rightarrow A$ when for all V, $(X, J, V) \vDash \Gamma \Rightarrow A$.

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Interpreting $x \Vdash JA$ as $\exists y (y, x) \in R \; y \Vdash A$, we can develop a Kripke semantics for the language and since Kripke frames are examples of modal spaces, this semantics is a special kind of topological semantics.

Some Classes of Modal Spaces

Definition

(i) The class **MS** consists of all modal spaces.

(ii) A modal space is called semi-cotemporal if $Ja = 0$ implies $a = 0$. Denote the set of these spaces by **sCoTS**.

(iii) A modal space is called temporal if $J(a) \leq a$. Denote the set of these spaces by **TS**.

(iv) A modal space is called cotemporal if $a \leq J(a)$. Denote the set of these spaces by **CoTS**.

Moreover, by **sS** we mean **sCoTS** $\cap$ **TS** and by **S** we mean **TS** $\cap$ **T**.

Topological/Kripke Semantics

Soundness-Completeness Theorem

(*i*) $\Gamma \vdash_{\mathbf{mJ}} A$ iff $\mathbf{MS} \vDash \Gamma \Rightarrow A$ iff $\Gamma \Rightarrow A$ is valid in all Kripke models.

(*ii*) $\Gamma \vdash_{\mathbf{sCoJ}} A$ iff $\mathbf{sCoTS} \vDash \Gamma \Rightarrow A$ iff $\Gamma \Rightarrow A$ is valid in all serial Kripke models.

(*iii*) $\Gamma \vdash_{\mathbf{CoJ}} A$ iff $\mathbf{CoTS} \vDash \Gamma \Rightarrow A$ iff $\Gamma \Rightarrow A$ is valid in all reflexive Kripke models.

(*iv*) $\Gamma \vdash_{\mathbf{J}} A$ iff $\mathbf{TS} \vDash \Gamma \Rightarrow A$ iff $\Gamma \Rightarrow A$ is valid in all transitive persistent Kripke models.

(*v*) $\Gamma \vdash_{\mathbf{sI}} A$ iff $\mathbf{sS} \vDash \Gamma \Rightarrow A$ iff $\Gamma \Rightarrow A$ is valid in all transitive serial persistent Kripke models.

(*vi*) $\Gamma \vdash_{\mathbf{IPC}} A$ iff $\mathbf{S} \vDash \Gamma \Rightarrow A$ iff $\Gamma \Rightarrow A$ is valid in all transitive reflexive persistent Kripke models.

Embedding Intuitionistic Implication into Predicative Ones

Theorem

Let (X, J) be a modal space and define $\Box a = 1 \to a$. Then the set $J\Box X$ is a Heyting algebra.

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Definition

Define the translation $(-)^j : \mathcal{L} \to \mathcal{L}_J$ as the following:

- (*i*) $p^j = J\Box p$, $\bot^j = \bot$ and $\top^j = J\top$.
- (*ii*) $(A \wedge B)^j = J\Box(A^j \wedge B^j)$.
- (*iii*) $(A \vee B)^j = A^j \vee B^j$.
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Theorem

For any $A \in \mathcal{L}$, $\Gamma \vdash_{IPC} A$ iff $\Gamma^j \vdash_{mJ} A^j$.

The Categorical Counterpart

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Thank you for your attention!